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Imbedding Theorems for Elliptic and Parabolic Operators in the Space C

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It is well known that a uniformly elliptic expression of order $2m$ with Hölder continuous coefficients generates an elliptic differential operator in the space C whose domain of definition is strictly included between C^{2m} and $C^{2m-\varepsilon}$. By using an appropriate notion of derivatives of fractional order, we give a more precise description of the imbedding of this domain of definition. Analogous results are established for parabolic operators. © 1989 Academic Press, Inc.

1. ELLIPTIC OPERATORS

The differential expression

$$(-1)^m \sum_{|\alpha|=2m} a_\alpha(x)\, D_x^\alpha v(x) \qquad (x \in \mathbb{R}^n)$$

with Hölder continuous coefficients generates an elliptic operator A with domain of definition $\mathscr{D}(A)$ in $C(\mathbb{R}^n)$ is the usual uniform ellipticity condition is satisfied. More precisely (see [1]),

$$\mathscr{D}(A) = \{v = R(\lambda) f : f \in C\}, \qquad Av = f - \lambda R(\lambda) f,$$

$$[R(\lambda) f](x) = \int_{\mathbb{R}^n} G(x, y; \lambda)\, f(y)\, dy,$$

where $G(x, y; \lambda)$ is the fundamental solution of the resolvent equation which is defined for $\lambda \geqslant \lambda_0$ with λ_0 sufficiently large. As a matter of fact (see, e.g., [2, p. 316]), the inclusion $C^{2m}(\mathbb{R}^n) \subset \mathscr{D}(A)$ is strict. From estimates of the derivatives of $G(x, y; \lambda)$ (see, e.g., [3]) it follows that $\mathscr{D}(A) \subseteq C^{2m-\varepsilon}(\mathbb{R}^n)$ for any $\varepsilon \in (0, 1]$. It turns out that $\mathscr{D}(A)$ is imbedded in a class of functions whose highest derivatives (i.e., of order $2m$) have a certain fractional degree of smoothness.

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One may associate with each multi-index $\alpha=(\alpha_1, ..., \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n = 2m$, where the α_k's are not necessarily entire, a Banach space $\tilde{C}^\alpha(\mathbb{R}^n)$ with norm

$$\|v\|_{\tilde{C}^\alpha} = \|v\|_C + \sup_{\substack{x \in \mathbb{R}^n \\ h_k > 0}} \left| \Delta_1(\gamma_1, h_1) \cdots \Delta_n(\gamma_n, h_n) \frac{\partial^{|\beta|} v(x)}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}} \right|;$$

here β_k is the entire part of α_k, γ_k is the fractional part of α_k, and

$$\Delta_k(\gamma_k, h_k)\, \varphi(x) = \begin{cases} [\varphi(x+h_k e_k) - \varphi(x)]\, h_k^{-\gamma_k} & (\gamma_k > 0), \\ \varphi(x) & (\gamma_k = 0), \end{cases}$$

with e_k being the kth unit vector in $\mathbb{R}^n$.

THEOREM 1. *$\mathscr{D}(A)$ is continuously imbedded in $\tilde{C}^\alpha(\mathbb{R}^n)$.*

Apart from $\tilde{C}^\alpha(\mathbb{R}^n)$, one may define another Banach space $\hat{C}^\alpha(\mathbb{R}^n)$ $(|\alpha| = 2m)$ by means of the norm

$$\|v\|_{\hat{C}^\alpha} = \|v\|_C + \sup_{\substack{x \in \mathbb{R}^n \\ k_k > 0 \\ |\beta| = 2m-1 \\ \beta_k \leqslant \alpha_k}} \left| \Delta_k^2(h_k) \frac{\partial^{|\beta|} v(x)}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}} \right|;$$

where now the α_k's are entire and $\Delta_k^2(h_k)\, \varphi(x) = [\varphi(x+h_k e_k) - 2\varphi(x) + \varphi(x-h_k e_k)]\, h_k^{-1}$.

THEOREM 2. *$\mathscr{D}(A)$ is continuously imbedded in $\hat{C}^\alpha(\mathbb{R}^n)$.*

2. PARABLIC OPERATORS

The differential expression

$$\frac{\partial v}{dt} + (-1)^m \sum_{|\alpha| = 2m} a_\alpha(t, x)\, D_x^\alpha(t, x) \qquad ((t, x) + \mathbb{R}^{n+1})$$

with Hölder continuous coefficients generates a parabolic operator $\sqcap(A)$ with domain of definition $\mathscr{D}[\sqcap(A)]$ in $C(\mathbb{R}^{n+1})$ if the usual uniform parabolicity condition is satisfied (see [1]). As a matter of fact, the inclusion $C_{t,x}^{1,2m}(\mathbb{R}^{n+1}) \subset \mathscr{D}[\sqcap(A)]$ is again strict. With each multi-index $\alpha = (\alpha_0, \alpha_1, ..., \alpha_n)$, $2m\alpha_0 + \alpha_1 + \cdots + \alpha_n = 2m$, one may associate Banach spaces $\tilde{C}^\alpha(\mathbb{R}^{n+1})$ and $\hat{C}^\alpha(\mathbb{R}^{n+1})$ in rather the same way as above, where the α_k's are not necessarily entire in the case $\tilde{C}^\alpha$, but entire in the case $\hat{C}^\alpha$.

THEOREM 3. *$\mathscr{D}[\sqcap(A)]$ is continuously imbedded in $\tilde{C}^{\alpha}(\mathbb{R}^{n+1})$.*

THEOREM 4. *$\mathscr{D}[\sqcap(A)]$ is continuously imbedded in $\hat{C}^{\alpha}(\mathbb{R}^{n+1})$.*

We point out that Theorems 1–4 hold for general elliptic respectively parabolic operators as well, since adding lower-order terms with continuous bounded coefficients does not affect the domain of definition of the operator.

In the remaining sections we shal sketch the proofs of our statements. From [1] it follows that all imbeddings are strict.

3. SOME AUXILIARY RESULTS

Consider the differential operators

$$A_k = (-1)^m \left(\frac{\partial}{\partial x_k}\right)^{2m} \qquad (k = 1, ..., n)$$

in $C(\mathbb{R}^n)$ which are defined on $C^{0, ..., 2m, ..., 0}_{x_1, ..., x_k, ..., x_n}(\mathbb{R}^n)$ and satisfy the estimate

$$\|(\partial/\partial x_k)^r(\lambda + A_k)^{-1}\|_{C \to C} \leqslant M\lambda^{r/2m-1} \tag{1}$$

for $r = 0, 1, ..., 2m$ and $\lambda > 0$. Let the space $E_\rho = E_\rho(C, A_k)$ $(0 < \rho < 1;$ $k = 1, ..., n)$ be defined by the norm

$$\|V\|_{E_\rho} = \|v\|_C + \sup_{\lambda > 0} \|\lambda^\rho A_k(\lambda + A_k)^{-1}v\|_C.$$

LEMMA 1 (See, e.g., [4]). *The formula*

$$E_\rho = E_\rho(C, A_k) = \begin{cases} \tilde{C}^{\alpha}(\mathbb{R}^n) & \text{if } 2m\rho \notin \mathbb{N}, \\ \hat{C}^{\alpha}(\mathbb{R}^n) & \text{if } 2m\rho \in \mathbb{N}, \end{cases}$$

holds, where $\alpha_k = 2m\rho$ and $\alpha_i = 0$ for $i \neq k$.

LEMMA 2. *The estimate*

$$\|(\partial/\partial x_k)^r(\lambda + A_k)^{-1}\|_{E_\rho \to C} \leqslant M\lambda^{r/2m-\rho-1} \tag{2}$$

holds for $r = 1, ..., 2m$, $0 < \rho < r/2m$, and $\lambda > 0$.

We remark that analogous results are true for the operators A_k in $C(\mathbb{R}^{n+1})$, the operator $A_0 = \partial/\partial t$ which is defined on $C^{1,0,...,0}_{t,x_1,...,x_n}(\mathbb{R}^{n+1})$, and the operators $B_0 = -A_0^2$. Denote by $\sqcap_0$ the operator defined by the differential expression

$$\frac{\partial v}{dt} + (-1)^m \sum_{r=1}^{2m} \left(\frac{\partial}{\partial x_r}\right)^{2m} v + v$$

in $C(\mathbb{R}^{n+1})$.

LEMMA 3. *The estimate*

$$\|D_x^{2m-k} v\|_{E_{k/2m}(C, A_0)} \leqslant M \|\sqcap_0 v\|_C \tag{3}$$

holds for $k = 1, ..., 2m-1$ *and* $v \in \mathscr{D}(\sqcap_0)$.

One may replace the operators A and $\sqcap(A)$ in Theorems 1–4 by $A + \lambda_0$ and $\sqcap(A) + \lambda_0$, respectively, for sufficiently large λ_0.

LEMMA 4 (See, e.g., [3–5]). *The estimates*

$$\begin{gathered} \|(\lambda + A)^{-1}\|_{C \to C^r} \leqslant M(r)(\lambda + 1)^{r/2m - 1}, \\ \|(\lambda + A)^{-1}\|_{C^\varepsilon \to C^{2m+\varepsilon}} \leqslant M(\varepsilon), \end{gathered} \tag{4}$$

and

$$\begin{gathered} \|[\lambda + \sqcap(A)]^{-1}\|_{C \to C_{t,x}^{r/2m, r}} \leqslant M(r)(\lambda + 1)^{r/2m - 1}, \\ \|[\lambda + \sqcap(A)]^{-1}\|_{C_{t,\alpha}^{\varepsilon/2m, \varepsilon} \to C_{t,\alpha}^{1+\varepsilon/2m, 2m+\varepsilon}} \leqslant M(\varepsilon) \end{gathered} \tag{5}$$

hold for $r \in [0, 2m)$, $\varepsilon \in (0, \varepsilon_0]$, *and* $\lambda \geqslant 0$.

4. PROOF OF THEOREMS 1 AND 2

We may assume in Theorem 1 that $\alpha_1, ..., \alpha_l$ are non-entire, $2 \leqslant l \leqslant 2m$, $\alpha_1 + \cdots + \alpha_l = k$, $1 \leqslant k \leqslant 2m$. By Lemma 1, it suffices to show that the function

$$\begin{aligned} &\psi(\lambda_1, ..., \lambda_l) \\ &\quad = \lambda_1^{\alpha_1/2m} \cdots \lambda_l^{\alpha_l/2m} \|A_1(\lambda_1 + A_1)^{-1} \cdots A_l(\lambda_l + A_l)^{-1} D_x^{2m-k} A^{-1}\|_{C \to C} \end{aligned}$$

is bounded. Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant l_l$. Since the A_i commute, we have to show only, by (1), the boundedness of

$$\lambda_1^{\alpha_1/2m} \lambda_2^{(k-\alpha_1)/2m} \|A_1(\lambda_1 + A_1)^{-1} A_2(\lambda_2 + A_2)^{-1} D_x^{2m-k} A^{-1}\|_{C \to C}.$$

The formula $A^{-1}=\int_0^\infty (\tau+A)^{-2}\,d\tau$, in turn, reduces the problem to estimating the integral

$$J(\lambda_1, \lambda_2)=\lambda_1^{\alpha_1/2m}\lambda_2^{(k-\alpha_1)/2m}\int_0^\infty \varphi(\lambda_1, \lambda_2; \tau)\,d\tau,$$

where

$$\begin{aligned}\varphi(\lambda_1, \lambda_2; \tau)&\\ &=\|A_1(\lambda_1+A_1)^{-1}A_2(\lambda_2+A_2)^{-1}D_x^{2m-k}(\tau+A)^{-2}\|_{C\to C}\\ &=\|(\partial/\partial x_1)^{2m-k}(\lambda_1+A_1)^{-1}A_2(\lambda_2+A_2)^{-1}D_x^{2m}(\tau+A)^{-1}\|_{C\to C}. \qquad (6)\end{aligned}$$

Now, (1) and (4) imply that $\varphi\leqslant M\tau^{-1-k/2m}$, (6) implies that $\varphi\leqslant M\lambda_1^{-k/2m}\tau^{-1}$, and (2) implies that $\varphi\leqslant M\lambda_1^{-k/2m}\lambda_2^{-\varepsilon/2m}\tau^{\varepsilon/2m-1}$. Applying these estimates on the intervals $[\lambda_1, \infty)$, (λ_2, λ_1), and $[0, \lambda_2]$, respectively, we conclude that

$$J(\lambda_1, \lambda_2)\leqslant M\left(\frac{\lambda_2}{\lambda_1}\right)^{(k-\alpha_1)/2m}\left[\frac{2m}{k}+\log\frac{\lambda_1}{\lambda_2}+\frac{2m}{\varepsilon}\right],$$

hence

$$\psi(\lambda_1, ..., \lambda_2)\leqslant M\max_{i=1,...,l}\sup_{t\geqslant 0} t^{(\alpha_i-k)/2m}(1+\log t)<\infty.$$

In order to prove Theorem 2, one has to estimate the integrals

$$J_i(\lambda)=\lambda^{1/2m}\int_0^\infty \varphi_i(\lambda, \tau)\,d\tau,$$

where

$$\begin{aligned}\varphi_i(\lambda, \tau)&=\|A_i(\lambda+A_i)^{-1}D_x^{2m-1}(\tau+A)^{-2}\|_{C\to C}\\ &=\|(\partial/\partial x_i)^{2m-1}(\lambda+A_i)^{-1}D_x^{2m}(\tau+A)^{-2}\|_{C\to C}.\end{aligned}$$

Here it follows from (1) and (4) that $\varphi_i\leqslant M\tau^{-1-1/2m}$ and from (2) and (4) that $\varphi_i\leqslant M\lambda^{-(1+\varepsilon)/2m}\tau^{\varepsilon/2m-1}$. These estimates lead to the desired inequality

$$J_i(\lambda)\leqslant M\cdot 2m+M\frac{2m}{\varepsilon}<\infty.$$

5. Proof of Theorems 3 and 4

The proof of Theorem 3 follows essentially the same line as that of Theorem 1. The difference consists in estimating the expression

$$\varphi_i(\lambda_0, \lambda_i) = \|A_0(\lambda_0 + A_0)^{-1} A_i(\lambda_i + A_i)^{-1} D_x^{2m-k}[\tau + \sqcap(A)]^{-2}\|_{C \to C}$$

for $\lambda_0 \geqslant \lambda_i$ $(i = 1, ..., 2m;\ k = 1, ..., 2m - 1)$. Here one need not extract k derivatives of A_0 with respect to x, byt just apply Lemma 3 directly.

Similarly, Theorem 4 is proved in rather the same way as Theorem 2, with additional estimates for the integral

$$\lambda^{1/2} \int_0^\infty \|B_0(\lambda + B_0)^{-1}[\tau + \sqcap(A)]^{-2}\|_{C \to C}\, d\tau$$

which are obtained by using Lemma 4.

Finally, we remark that analogues to Theorems 1–4 may be established for elliptic and parabolic operators in the spaces $L_1(\mathbb{R}^n)$ and $L_1(\mathbb{R}^{n+1})$, respectively; details will be given in a forthcoming paper.

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